JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 137, 99–121 (1989)

A Borsuk–Ulam Theorem for Orthogonal T^k and Z_p^r Actions and Applications

WACŁAW MARZANTOWICZ

Institute of Mathematics, University of Gdańsk
Wita Stwosza 57, 80-952 Gdańsk, Poland

Submitted by Ky Fan

Received December 10, 1986

A version if the Borsuk–Ulam theorem is proved for the torus T^k and p-torus Z_p^r. We deal with the case of a nonempty fixed point set of groups on the sphere of representation. As an application, a geometrical index theory is introduced for these groups. The index is used to obtain the existence of multiple critical points for functions invariant under T^k or Z_p^r action. © 1989 Academic Press, Inc.

1. INTRODUCTION

In order to formulate our main result we introduce the following notation. Let G be a compact Lie group. We denote by G_0 the component of identity of G and by Γ the quotient group G/G_0. We use standard notation from the theory of compact transformation groups (see for instance [4] or [7]). In particular, for every subgroup $H \subset G$, the fixed point set of H on a G-space X is denoted by X^H. Also, for a G-equivariant map $f: X \to Y$ between two G-spaces, we denote by f^H its restriction to the space X^H. The symbol (n, m) stands for the greatest common divisor of integers n, m and $|G|$ stands for the order of the (finite) group G. We use the symbol $S(V)$ to denote the unit sphere of an orthogonal representation V of G.

We will work with the following definition of the Borsuk–Ulam property.

DEFINITION I. (A) We say that G has the Borsuk–Ulam property in the weak sense A if for every orthogonal representation V of G and every G-equivariant map $f: S(V) \to S(V)$ such that

$$(\deg f^G, |\Gamma|) = 1$$

if $\dim V^G \geqslant 1$ and $\deg f^G \neq 0$, we have

$$\deg f \neq 0.$$

0022-247X/89 $3.00

(Note that in case $V^G = \{0\}$ there is no condition on $\deg f^G$ and the property then requires that $\deg f \neq 0$ for every G-equivariant map. Also, if $G = G_0$ then the condition $(\deg f^G, |\Gamma|) = 1$ means that $\deg f^G \neq 0$.)

(B) We say that G has the Borsuk–Ulam property in the weak sense B if for every pair of orthogonal representations $W \subsetneq V$ of G, with $V^G = \{0\}$, there is no G-equivariant map

$$f\colon S(V) \to S(W).$$

DEFINITION II. (A) We say that G has the Borsuk–Ulam property in the strong sense A if for every pair of orthogonal representations V, W of G, with $\dim W = \dim V$, $\dim W^G = \dim V^G$, and every G-equivariant map

$$f\colon S(V) \to S(W)$$

such that

$$(\deg f^G, |\Gamma|) = 1$$

if $\dim V^G \geqslant 1$ and $\deg f^G \neq 0$, we have

$$\deg f \neq 0.$$

(If $W^G = \{0\}$ or $G = G_0$, the same remark as above applies.)

(B) We say that G has the Borsul–Ulam property in the strong sense B if there is no G-equivariant map $f\colon S(V) \to S(W)$ where W is an orthogonal representation of G such that $W \subsetneq U$ for an orthogonal representation U, $\dim U = \dim V$ and $U^G = \{0\}$.

To shorten notation, we say that G has property I.A (respectively I.B, II.A, and II.B) if G has to Borsuk–Ulam property in the sense of Definition I.A (respectively Def. I.B, Def. II.B).

We now state a simple observation we shall frequently use.

1.1. PROPOSITION. *With the above notation we have the following implications*:

$$\begin{array}{ccc} G \text{ has II.A} & \Longrightarrow & G \text{ has II.B} \\ \Big\Downarrow & & \Big\Downarrow \\ G \text{ has I.A} & \Longrightarrow & G \text{ has I.B.} \end{array}$$

Proof. Indeed, as in the classical Z_2-case, composing f with the inclusion of $S(W)$ into $S(U)$ (or into $S(V)$ in the case of property I) we get a G-equivariant map from $S(V)$ into $S(U)$ (or into $S(V)$) of degree 0, which gives the horizontal implications. The vertical implications are evident.

We shall use the symbol T^k to denote the k-dimensional torus $S^1 \times \cdots \times S^1$ and Z_p^r for the p-torus $Z_p \times \cdots \times Z_p$, p-prime.

Our main result formulates as follows.

THEOREM 1. *Suppose that V and W are orthogonal representations of the torus T^k, $k \geqslant 1$, such that* $\dim W = \dim V$ *and* $\dim W^{T^k} = \dim V^{T^k} \geqslant 1$.

Then for every T^k-equivariant map $f\colon S(V) \to S(W)$, $\deg f = 0$ *if and only if* $\deg f^{T^k} = 0$.

If in addition, $\dim W^{T^k} = \dim V^{T^k} = 0$, *then* $\deg f \neq 0$ *for every T^k-equivariant map $f\colon S(V) \to S(W)$.*

THEOREM 2. *Suppose that V and W are orthogonal representations of Z_p^r, $r \geqslant 1$, such that* $\dim W = \dim V$ *and* $\dim W^{Z_p^r} = \dim V^{Z_p^r} \geqslant 1$. *Then for every Z_p^r-equivariant map $f\colon S(V) \to S(W)$*

$$\deg f \equiv 0 \pmod p$$

if and only if

$$\deg f^{Z_p} \equiv 0 \pmod p.$$

If, in addition, $\dim W^{Z_p^r} = \dim V^{Z_p^r} = 0$, *then* $\deg f \not\equiv 0 \pmod p$, *for every Z_p^r-equivariant map $f\colon S(V) \to S(W)$.*

In other words, Theorems 1 and 2 say that the groups T^k and Z_p^r have property II.A.

In our terminology, the classical Borsul–Ulam theorem states that $G = Z_2$ has property I.A, hence also I.B, if $V^G = \{0\}$ [5]. Observe that since Z_2 has only one nontrivial irreducible representation, property I.B coincides with property II.B and I.A with II.A. It is known and not difficult to check that Z_2 has property I.A.

The case $G = Z_p$, p prime, was studied by many authors. For example, in [15] it is shown that

$$\deg f = \deg f^{Z_p} \pmod p$$

for every Z_p-equivariant map $f\colon S(V) \to S(W)$ as in Definition II.A.

From the results of T. tom Dieck (see [7, V] for the complete bibliography of Dieck's results on this subject) or from the results of [16] it follows that any p-group, p-prime, has property I.A., Indeed, from the mentioned result follows that for a G-equivariant map f of the sphere $S(V)$ we have the congruence of the Lefschetz number

$$L(f) \equiv L(f^G) \pmod{p}$$

for a G-equivariant map of the sphere of an orthogonal representation of a p-group.

The case $G = S^1$ was extensively studied by several authors with a view to its applications to nonlinear analysis. In [16] it is shown that the torus T^k has property I.A. Using analytic methods, L. Nirenberg proved that $G = S^1$ has property II.A [17]. This was also proved independently by E. Fadell, S. Husseini, and P. Rabinowitz. They use the relative cohomological S^1-index introduced by them in [13]. It is worth pointing our that for $G = S^1$ the above result is covered by an earlier, unfortunately unpublished, result of P. Traczyk. In 1977 he showed in his M.Sc. work that S^1 has property II.A. Moreover his simple geometrical method of proof gives also, as in [17], a relation between the characters of representations V and W for which a G-map $f\colon S(V) \to S(W)$ exists. Other results of [19] are published in [20].

The paper is divided into two parts. In the next section Theorems 1 and 2 are proved. In the last section we give a new definition of a geometrical G-index of a pair of metric G-spaces in the cases $G = T^k$ and $G = Z_p^r$. We show that this invariant has the required properties analogous to the properties of the Z_2-genus and the geometrical S^1-index. This allows us to apply the minimax method to obtain a lower bound of the number of critical values of a G-invariant smooth function. The construction of the G-index is based on the Borsuk–Ulam theorem for the torus and the p-torus.

2. The Borsuk–Ulam Theorem for the Group $G = T^k$ and $G = Z_p^r$

Section 2 is devoted to the study of G-equivariant maps between the spheres of orthogonal representations for $G = T^k$, Z_p^r. Our purpose is to prove Theorem 1 and Theorem 2. The main idea is to reduce the groups to S^1 and Z_p, respectively, by an induction argument. This will be done by restricting the action to subgroups isomorphic to T^{k-1} and Z_p^{r-1}, respectively. The proof of the Borsuk–Ulam theorem for S^1 and Z_p is adapted from the unpblished paper of Traczyk [19]. In the case $G = S^1$ it is almost literally taken from there. The basic geometric ingredient is the fact that the join of spheres of two orthogonal representations is the sphere in the direct sum of those representations and the degree of the join of two maps between spheres is the product of the degrees of these maps.

Let us recall the notion of join of two topological spaces. For given paracompact G-spaces X and Y, by the join $X * Y$ we mean the set of all pairs $(tx, (1-t)y)$, $x \in X$, $y \in Y$, $t \in [0, 1]$ with the identifications $(0x, 1y) \sim (0x', 1y)$ and $(1x, 0y) \sim (1x, 0y')$ equipped with a suitable topology. If X and Y are G-spaces then $X * Y$ has a natural G-structure given by

$$g(tx, (1-t)y) = (tgx, (1-t)gy).$$

For given maps $f: X \to X_1$ and $h: Y \to Y_1$ we can define the join $f * h: X * Y \to X_1 * Y_1$ which is G-equivariant if f and h are G-equivariant. We shall frequently use the following well-known fact.

2.1. For any G-orthogonal representations V, W, V_1, W_1 with $\dim V_1 = \dim V$, $\dim W_1 = \dim W$, and G-equivariant maps $f: S(V) \to S(V_1)$ and $h: S(W) \to S(W_1)$,

$$S(V) * S(W) \text{ is } G\text{-homeomorphic to } S(V \oplus W)$$

and $\deg(f * h) = \deg f \cdot \deg h$.

Here and subsequently, $RO^+(G)$ ($RU^+(G)$) stands for the semiring of all real orthogonal (respectively, complex unitary) representations of a given compact Lie group G. For $V \in RO^+(G)$ (or $RU^+(G)$), we shall denote by V_G the complementing factor to the fixed point subspace $V^G \subset V$. V_G is a direct sum of nontrivial irreducible representations of G.

We have to use the following statements of Z_p-equivariant maps between the spheres of orthogonal representations of Z_p, p-prime [15, Cor. 2.4].

2.2. Lemma. *Suppose that $V, W \in RO(Z_p)$ are such that* $\dim W = \dim V$ *and* $\dim W^{Z_p} = \dim V^{Z_p} \geqslant 1$. *Then for any two Z_p-equivariant maps $f, h: S(V) \to S(W)$, if* $\deg f^{Z_p} \equiv \deg h^{Z_p} \pmod p$ *then* $\deg f \equiv \deg h \pmod p$.

2.3. Lemma. *Suppose that $V \in RO^+(Z_p)$,* $\dim V^{Z_p} \geqslant 1$. *Then for every Z_p-equivariant map $f: S(V) \to S(V)$ we have* $\deg f \equiv \deg f^{Z_p} \pmod p$.

Remark. Originally in [8] these facts were stated under the assumption that $\dim V^{Z_p} = \dim W^{Z_p} \geqslant 2$ (or $\dim V^{Z_p} \geqslant 2$). With the convention that the degree of a map of the zero-dimensional sphere S^0 is equal to 1, 0. or -1 according to whether the map is the identity, the mapping into one point, or the permutation of the points of S^0, we can easily get rid of this assumption. In fact Lemma 2.2 and Lemma 2.3 are equivalent [15], and the statement of Lemma 2.3 follows from the congruence for the Lefschetz numbers of a Z_p-equivariant map [7, 16].

In the next step, we use Lemma 2.2 to get its analogue for $G = S^1$ [19].

2.4. LEMMA. *Assume that* $V, W \in RO^+(S^1)$ *are such that* $\dim V = \dim W$ *and* $\dim W^{S^1} = \dim V^{S^1} \geqslant 1$. *Suppose that* $f, h\colon S(V) \to S(W)$ *are* G*-equivariant maps with* $\deg f^{S^1} = \deg h^{S^1}$. *Then*

$$\deg f = \deg h.$$

Proof. First observe that there exists $n_0 \in N$ such that

$$S(V)^{Z_p} = S(V)^{S^1} \qquad \text{and} \qquad S(W)^{Z_p} = S(W)^{S^1}$$

for every prime $p > n_0$. For such p, we have

$$\deg f^{Z_p} = \deg f^{S^1} \qquad \text{and} \qquad \deg h^{Z_p} = \deg h^{S^1},$$

consequently $\deg f^{Z_p} = \deg h^{Z_p}$, and from (2.2) it follows that

$$\deg f \equiv \deg h \pmod p. \tag{$*$}$$

Since $(*)$ holds for every $p > n_0$,

$$\deg f = \deg h,$$

and the proof is complete.

As a direct consequence of Lemma 2.4 we get the following proposition [19].

2.5. PROPOSITION. *Suppose* $V, W \in RO^+(S^1)$ *are such that* $\dim W = \dim V$ *and* $\dim W^{S^1} = \dim V^{S^1} = 0$.

Then for all S^1*-equivariant maps*

$$f, h\colon S(V) \to S(W)$$

we have $\deg f = \deg h$.

Proof. Let us denote by V^0 the one-dimensional trivial representation of S^1 and by $\mathbf{1}$ the identity map on $S(V^0)$. Taking the joins $f * \mathbf{1}$ and $h * \mathbf{1}$ we form two S^1-equivariant maps from $S(V \oplus V^0) = S(V) * S(V^0)$ into $S(W \oplus V^0) = S(W) * S(V^0)$. We have $\deg(f * \mathbf{1}) = \deg f$, $\deg(h * \mathbf{1}) = \deg h$, and $\deg(f * \mathbf{1})^{S^1} = \deg(h * \mathbf{1})^{S^1} = \deg \mathbf{1} = 1$, which yields the proposition by Lemma 2.2.

By the same argument as above, we can deduce from Lemma 2.2 the following statement.

2.6. PROPOSITION. *Suppose that* $V, W \in RO^+(Z_p)$ *are such that*

$\dim W = \dim V$ *and* $\dim W^{Z_p} = \dim V^{Z_p} = 0$. *Then for all* Z_p*-equivariant maps* $f, h \colon S(V) \to S(W)$ *we have*

$$\deg f \equiv \deg h \pmod p.$$

Now we shall study the degree of an S^1-equivariant map $f \colon S(V) \to S(W)$ for fixed V, W and the value of $\deg f^{S^1}$. We begin with some notation. We denote by $\tilde{V}^n$ the one-dimensional complex representation of S^1 given by the formula

$$\rho(s)v = s^n v$$

for every $v \in \tilde{V}^n$ and $s \in S^1 \subset \mathbb{C}$. By $V^n = r\tilde{V}^n$ we denote the two-dimensional real orthogonal representation of S^1 which is obtained from V^n by the restriction of the scalar field V^n and is an irreducible representation if $n \neq 0$; and $\{V^n\}_{n \geqslant 1}$ together with V^0 form the complete list of all irreducible representations of S^1 up to isomorphism. Hence every $V \in RO(S^1)$ can be written as

$$V = V^{S^1} \oplus n_1 V^{i_1} \oplus \cdots \oplus n_k V^{i_k},$$

or shortly

$$V = V^{S^1} \oplus V_{S^1}$$

in our notation.

Moreover the sequence

$$\dim V^{S^1}, \underbrace{i_1, \ldots, i_1}_{n_1}, \ldots, \underbrace{i_k, \ldots, i_k}_{n_k}$$

determines V up to isomorphism.

The next theorem expresses the relation between the degree of an S^1-equivariant map $f \colon S(V) \to S(W)$ and the degree of its restriction f^{S^1} to the fixed point set [19].

2.7. THEOREM (L. Nirenberg, P. Traczyk). *Suppose that*

$$V, W \in RO^+(S^1),$$
$$V = V^{S^1} \oplus n_1 V^{i_1} \oplus \cdots \oplus n_k V^{i_k},$$
$$W = V^{S^1} \oplus n'_1 V^{i'_1} \oplus \cdots \oplus n'_m V^{i'_m},$$

are such that $\dim W = \dim V$ *and* $\dim W^{S^1} = \dim V^{S^1}$. *Then for every* S^1*-equivariant map*

$$f \colon S(V) \to S(W)$$

we have

$$\deg f = T(V, W) \deg f^{S^1},$$

where $T(V, W) = i_1^{-n_1} i_2^{-n_2} \cdots i_k^{-n_k} i_1'^{n_1'} \cdots i_m'^{n_m'}$ *depends on the representations* V, W *only.*

Proof. We consider the pair of representations

$$\begin{aligned} V &= V^{S^1} \oplus \tfrac{1}{2} \dim V_{S^1} \cdot V^1 \oplus V_{S^1} \\ &= V^{S^1} \oplus \tfrac{1}{2} \dim V_{S^1} \cdot V^1 \oplus n_1 V^1 \oplus \cdots \oplus n_k V^{i_k}, \\ W &= W^{S^1} \oplus V_{S^1} \oplus W_{S^1} = W^{S^1} \oplus n_1 V^{i_1} \oplus \cdots \oplus n_k V^{i_k} \\ &\quad \oplus n_1' V^{i_1'} \oplus \cdots \oplus n_m' V^{i_m'}. \end{aligned}$$

It is easy to check that $z \mapsto z^k$ gives an S^1-map

$$f_{n,kn}\colon S(V^n) \to S(V^{kn})$$

of degree k.

Using these maps, we can form an S^1-map

$$h\colon S(\tfrac{1}{2} \dim V_{S^1} \cdot V^1) \to S(n_1' V^{i_1'} \oplus \cdots \oplus n_m' V^{i_m'})$$

which is defined to be the join

$$\underbrace{f_{1,i_1'} * \cdots * f_{1,i_1'}}_{n_1'} * \cdots * \underbrace{f_{1,i_m'} * \cdots * f_{1,i_m'}}_{n_1'}.$$

Finally we consider the S^1-map

$$f^{S^1} * h * \mathbf{1}\colon S(V) \to S(W)$$

of degree $\deg f^{S^1} \cdot i_2'^{n_1'} \cdot i_2'^{n_2'} \cdot \cdots \cdot i_m'^{n_m'}$.

On the other hand we can analogously construct an S^1-map

$$h'\colon S(\tfrac{1}{2} \dim V_{S^1} \cdot V^1) \to S(n_1 V^{i_1} \oplus \cdots \oplus n_k V^{i_k})$$

and consequently the S^1-map

$$f * h'\colon S(V) \to S(W).$$

We have $\deg(f * h') = \deg f \cdot i_1^{n_1} \cdot \cdots \cdot i_k^{n_k}$ and $(f^{S^1} * h * \mathbf{1})^{S^1} = f^{S^1}$, $(f * h')^{S^1} = f^{S^1}$.

On account of Proposition 2.5 applied to the S^1-maps $f^{S^1} * h * \mathbf{1}$ and $f * h'$ we have

$$\deg f \cdot i_1^{n_1} \cdot i_2^{n_2} \cdot \cdots \cdot i_k^{n_k} = \deg f^{S^1} \cdot i_1'^{n_1'} \cdot i_2'^{n_2'} \cdot \cdots \cdot i_m'^{n_m'},$$

which proves the theorem.

2.8. Corollary. *Under the assumptions of Theorem* 2.7 *we have* $\deg f = 0$ *if and only if* $\deg f^{S^1} = 0$. *In our terminology, Corollary* 2.8 *says that* S^1 *has the Borsuk–Ulam property* II.A.

The next statement can be deduced from Theorem 2.7 by a suspension argument [19].

2.9. Proposition. *If in Theorem* 2.7 *we assume that*

$$\dim W^{S^1} = \dim V^{S^1} = 0,$$

then $\deg f = T(V, W) = i_1'^{n_1'} \cdot \cdots \cdot i_m'^{n_m'} \cdot i_1^{-n_1} \cdot \cdots \cdot i_k^{-n_k}$. *In particular,* $\deg f \neq 0$ *for every* S^1*-map* $f\colon S(V) \to S(W)$ *if* $V^{S^1} = W^{S^1} = \{0\}$.

We now turn to the case $G = Z_p$. Using a similar argument we will deal with Z_p-equivariant maps $f\colon S(V) \to S(W)$ with fixed $V, W \in RO^+(Z_p)$ and $\deg f^{Z_p} \pmod p$.

As before we need some notation. We have to consider the cases $G = Z_2$ and $G = Z_p$, p an odd prime, separately. Assume first that $G = Z_p$, p an odd prime. $\tilde{V}^n$, $n = 1, 2, ..., p-1$, denotes the one-dimensional complex representation of Z_p given by

$$\rho(g)v = g^n v,$$

for all $v \in \tilde{V}^n$ and $g \in Z_p \subset S^1 \subset C$.

Restricting in $\tilde{V}^n$ the scalar field to the real numbers we obtain the two-dimensional real orthogonal representation $r(\tilde{V}^n)$ of Z_p denoted by V^n. The representation V^n is $\mathbb{R}$-isomorphic to V^{p-n}, and the representations V^n, $n = 1, 2, ..., (p-1)/2$, together with the one-dimensional trivial representation V^0 are all irreducible real representations of Z_p. This means that every $V \in RO^+(Z_p)$ can be written as

$$V = V^{Z_p} \oplus n_1 V^1 \oplus \cdots \oplus n_{(p-1)/2} V^{(p-1)/2},$$

where $n_i \in N \cup \{0\}$. The sequence $\dim V^{Z_p}, n_1, n_2, ..., n_{(p-1)/2}$, determines V up to isomorphism.

The group $G = Z_2$ has only two nonisomorphic irreducible representations: the one-dimensional nontrivial representation, denoted by V^1 and given by $\rho(g)v = -v$ for $v \in V$, $e \neq g \in Z_2$, and the one-dimensional trivial representation. This means that every $V \in RO^+(Z_2)$ can be written as $V = V^{Z_2} \oplus n_1 V^1$ and the numbers $\dim V^{Z_2}$, n_1 characterize V.

We can now state the Z_p-analogues of (2.7), (2.8), and (2.9).

2.10. THEOREM. *Suppose that*

$$V, W \in RO^+(Z_p),$$

$$V = V^{Z_p} \oplus n_1 V^1 \oplus \cdots \oplus n_{(p-1)/2} V^{(p-1)/2},$$

$$W = W^{Z_p} \oplus n_1' V^1 \oplus \cdots \oplus n_{(p-1)/2}' V^{(p-1)/2},$$

are such that $\dim W = \dim V$ *and* $\dim W^{Z_p} = \dim V^{Z_p}$. *Then for every* Z_p*-equivariant map* $f\colon S(V) \to S(W)$,

$$\operatorname{def} f \equiv T_p(V, W) \deg f^{Z_p} (\operatorname{mod} p),$$

where

$$T_p(V, W) = 1^{-n_1} \cdot 2^{-n_2} \cdot \cdots \cdot \left(\frac{p-1}{2}\right)^{-n_{(p-1)/2}} \cdot 1^{n_1'} \cdot \left(\frac{p-1}{2}\right)^{n_{(p-1)/2}'}$$

is a unit in Z_p.

In particular, if $\deg f \equiv \deg f^{Z_p} (\operatorname{mod} p)$ *is a unit* Z_p *then* $T_p(V, W) \equiv 1 \pmod p$.

2.11. COROLLARY. *Under the assumption of Theorem* 2.10 (*or Lemma* 2.3) *we have* $\deg f \equiv 0 \pmod p$ *if and only if* $\deg f^{Z_p} \equiv 0 \pmod p$.

2.12. PROPOSITION. *If in Theorem* 2.10 *we assume that*

$$\dim V^{Z_p^r} = \dim W^{Z_p^r} = 0,$$

then

$$\deg f \equiv 1^{n_1'} \cdot 2^{n_1'} \cdot \cdots \cdot \left(\frac{p-1}{2}\right)^{n_{(p-1)/2}'} \cdot 1^{-n_1} \cdot \left(\frac{p-1}{2}\right)^{n_{(p-1)/2}} \pmod p.$$

In particular, in that case $\deg f \not\equiv 0 \pmod p$ *and consequently* $\deg f \neq 0$ *for every* Z_p*-equivariant map* $f\colon S(V) \to S(W)$.

Observe that Theorem 2.10 is an extension of Lemma 2.2 and that it gives a relation between those $V, W \in RO^+(Z_p)$ for which there exists at least one Z_p-map $f: S(V) \to S(W)$. In our terminology, Corollary 2.11 says that $G = Z_p$ has the Borsuk–Ulam property II.A.

Proof of (2.10), (2.11), (2.12). All these facts follow from Lemma 2.2 in the same way as (2.7), (2.8), and (2.9) follow from Lemma 2.4. We have only to replace equalities by congruences modulo p. Moreover, Corollary 2.11 can be deduced from Theorem 2.10 as well. For $G = Z_2$ we have $T_2(V, W) = 1$, and Theorem 2.10 reduces to Lemma 2.3.

We are now in a position to prove the S^1 and Z_p versions of the Borsuk–Ulam theorem.

2.13. THEOREM (E. Fadell, S. Husseini, P. Rabinowitz [13], L. Nirenberg [17], and P. Traczyk [19]). *Suppose that* $V, W \in RO^+(S^1)$ *are such that*

$$\dim W^{S^1} = \dim V^{S^1} \geqslant 1.$$

If there exists an S^1*-equivariant map*

$$f: S(V) \to S(W) \qquad \textit{with} \quad \deg f^{S^1} \neq 0$$

then $\dim V \leqslant \dim W$.

If we assume that $\dim W^{S^1} = 0$ *then* $\dim V \leqslant \dim W$ *provided there exists an* S^1*-map* $f: S(V) \to S(W)$.

Proof. Suppose by contradiction that $\dim V > \dim W$. Since all non-trivial irreducible real representations are two-dimensional, $\dim V - \dim W$ is an even number. Hence we can consider the representation

$$\tilde{W} = \tfrac{1}{2}(\dim V - \dim W) \cdot V^1$$

and form now a representation $U = W \oplus \tilde{W}$.

Composing f with the inclusion $i: S(W) \to S(U)$ we get an S^1-map $\tilde{f}: S(V) \to S(U)$ with $\tilde{f}^{S^1} = f^{S^1}$. As $\tilde{f}$ factors through a lower dimensional sphere, $\deg \tilde{f} = 0$. On the other hand, $\dim W^{S^1} = \dim U^{S^1} = \dim V^{S^1}$ and $\deg f^{S^1} \neq 0$, and the statement of Theorem 2.7 leads to a contradiction. This proves the theorem.

The following analogue of the last theorem has the same proof; we have only to use Theorem 2.10 instead of Theorem 2.7.

2.14. THEOREM. *Suppose that* $V, W \in RO^+(Z_p)$ *are such that*

$$\dim W^{Z_p} = \dim V^{Z_p} \geqslant 1.$$

If there exists a Z_p*-equivariant map* $f\colon S(V)\to S(W)$ *with* $\deg f^{Z_p}\not\equiv 0 \pmod p$ *then* $\dim V\leqslant \dim W$.

If we assume that $\dim W^{Z_p}=\dim V^{Z_p}=0$ *then* $\dim V\leqslant\dim W$ *provided there exists a* Z_p*-equivariant map* $f\colon S(V)\to S(W)$.

Our next goal is to prove Theorem 1 and Theorem 2. We shall reduce the cases $G=T^k$, $k\geqslant 2$, and $G=Z_p^r$, $r\geqslant 2$, to the already discussed $G=S^1$ and $G=Z_p$. We shall prove both theorems simultaneously. To shorten notation, we write G for T^k and Z_p^r in the proof.

Proof of Theorems 1 *and* 2. The proof is by induction on k (on r). For $k=1$ the statement of Theorem 1 is covered by the conclusion of Theorem 2.13. Similarly, for $r=1$ the statement of Theorem 2 is covered by the conclusion of Corollary 2.11 and Proposition 2.12.

Suppose that $f\colon S(V)\to S(W)$ is a G-equivariant map as in Theorem 1 (Theorem 2). Assume that there exists a subtorus $H\subset G$ of codimension 1 (a subgroup $H\subset G$ of index p) such that $\dim V^H=\dim W^H$. Since G is abelian, $S^1=G/H$ ($Z_p=G/H$) acts on V^H and W^H and

$$f^H\colon S(V^H)\to S(W^H)$$

is S^1-equivariant (Z_p-equivariant).

We also have $(V^H)^{G/H}=V^G$ and $(W^H)^{G/H}=W^G$. If $\dim W^H=0$ then, by induction, $\deg f\neq 0$ ($\deg f\not\equiv 0 \pmod p$) for f considered as an H-equivariant map. We have $\dim W^G\leqslant \dim W^H=0$, which leads us to the second part of the statements of Theorems 1 and 2. If $\dim W^H\geqslant 1$ then from Theorem 2.7 and Lemma 2.3 it follows that $\deg f^H=T(V^H,W^H)\cdot \deg(f^H)^{S^1}$ ($\deg f^H\equiv \deg(f)^{Z_p} \pmod p$) for the G/H-equivariant map f^H.

By induction $\deg f=0$ if and only if $\deg f^H=0$ ($\deg f\equiv \deg f^H \pmod p$), which proves Theorems 1 and 2 also in this case.

The above shows that it is sufficient to prove that there is at least one subtorus $H\subset G$ of codimension 1 (a subgroup $H=G$ of index p) such that $\dim W^H=\dim V^H$. Observe that $V^{H_1}\cap V^{H_2}=V^G$ and $W^{H_1}\cap W^{H_2}=W^G$ for any two such distinct subgroups H_1, H_2.

Every nontrivial irreducible representation V^α of G is given by a homomorphism $\rho_\alpha\colon G\to S^1$, so that $\ker\rho_\alpha$ contains a subtorus of codimension 1 (is a subgroup of index p). This means that the space V is spanned by subspaces V^H, where H runs through all subgroups as above; the same is true of W. Comparing the last two facts, we see that V/V^G is the direct sum of subspaces V^H/V^G; the same is true for W/W^G. Theorem 2.13 (or (2.14)) applied to the G/H-equivariant map $f^H\colon S(V^H)\to S(W^H)$ states that $\dim V^H\leqslant \dim W^H$, which gives

$$\dim(V^H/V^G)\leqslant \dim(W^H/W^G).$$

Finally, the inequality

$$\begin{aligned}\dim(V/V^G) &= \sum_H \dim(V^H/V^G)\\ &\leqslant \sum_H \dim(W^H/W^G) = \dim(W/W^G)\end{aligned}$$

shows that $\dim V^H = \dim W^H$ for every such H, since $\dim(V/V^G) = \dim(W/W^G)$. This completes the proof.

We conclude this section with the following version of the Borsuk–Ulam theorem for $G = T^k, Z_p^r$.

2.15. THEOREM. *Suppose $V, W \in RO^+(T^k)$, $k \geqslant 1$, are such that*

$$\dim V^{T^k} = \dim W^{T^k}.$$

Then if there exists a T^k-equivariant map

$$f: S(V) \to S(W) \qquad \textit{with} \quad \deg f^{T^k} \neq 0$$

then

$$\dim V \leqslant \dim W.$$

In particular, there is no T^k-equivariant map $f: S(V) \to S(W)$

$$\textit{if} \quad \dim V^{T^k} = \dim W^{T^k} = 0 \qquad \textit{and} \qquad \dim V > \dim W.$$

2.16. THEOREM. *Suppose that $V, W \in RO^+(Z_p^r)$ are such that*

$$\dim W^{Z_p^r} = \dim V^{Z_p^r}.$$

Then if there exists a Z_p^r-equivariant map

$$f: S(V) \to S(W) \qquad \textit{with} \quad \deg f^{Z_p^r} \not\equiv 0 \ (\mathrm{mod}\ p)$$

then

$$\dim V \leqslant \dim W.$$

In particular there is no Z_p^r-equivariant map

$$f: S(V) \to S(W) \qquad \textit{if} \quad \dim W^{Z_p^r} = \dim V^{Z_p^r} = 0$$

and

$$\dim V > \dim W.$$

Proof of Theorems 2.15 *and* 2.16. The conclusions follow from Theorem 1 and Theorem 2, respectively. The only important point is that every irreducible real nontrivial representation of $G = T^k$, Z_p^r is of dimension two. It follows that for $V, W \in RO^+(G)$ with $\dim W^G = \dim V^G$ and $\dim V > \dim W$, there exists a $\tilde{W} \in RO^+(G)$ such that $\tilde{W}^G = \{0\}$ and $\dim \tilde{W} = \dim V - \dim W$. This allows us to use Proposition 1.1 to deduce Theorems 2.15 and 2.16 from Theorems 1 and 2.

3. A Geometrical G-Index for the Torus and the p-Torus

Let M be a compact smooth G-manifold without boundary and $f: M \to R$ a C^1 function on M. We say that f is G-invariant if $f(gx) = f(x)$ for all $g \in G$ and $x \in M$.

Several authors observed that the symmetry of the studied problem allows one to use analogues of the Ljusternik–Schnirelman category method to obtain a lower bound for the number of critical values of a G-invariant function f [1–3, 6, 9–12, 18]. In order to apply a nonlinear minimax procedure, some invariants, called G-indexes and denoted ind_G or γ_G were constructed on the category of G-metric spaces. All constructions give invariants with the properties of the Ljusternik–Schnirelman category.

There are two distinct approaches to the construction of such a G-index of a G-space X.

The first is defined in terms of the cohomology ring $H^*(X \times_G EG; \kappa)$, κ a field, and called the cohomological G-index [12–14]. Although this construction is general and has all the required properties, it depends on the choice of an element $\alpha \in H^*(BG; \kappa)$ (or more generally on a choice of a subring $\Lambda \subset H^*(BG; \kappa)$) and can be canonically defined only in the cases $G = Z_2$, S^1, and S^3.

In the alternative approach, the G-index of a G-space X measures the obstruction to the existence of a G-mapping of X into the sphere of a unitary representation of G. It is called the geometrical G-index on X.

For $G = Z_2$, it was introduced, and called the Z_2-genus, in [6] and [18]. In 1981 V. Benci defined and applied the geometrical S^1-index of a fixed point free S^1-space X [2]. In order to obtain further interesting applications, the theory of the geometrical S^1-index was extended to the case $X^{S^1} \neq \emptyset$ in [3].

In this section we introduce a definition of the geometrical G-index of a pair of metric G-spaces in the cases $G = T^k$ and $G = Z_p$. The Borsuk–Ulam theorem lets us show that this G-index has the required normalization property. The remaining part of the construction is similar to the earlier definitions [2, 3]. We obtain the monotonicity of our G-index with respect to a G-map being a homotopy equivalence on the fixed point set of G (compare with [3] where such a G-map has to be the identity on this set). This is done by the use of some homotopical invariant of the fixed point set of G.

As in preceding sections, G denotes the k-dimensional torus T^k, $k \geqslant 1$, or the p-torus Z_p^r, $r \geqslant 1$. We use the symbol $\mathscr{F}$ to denote the category of all separable G-metric spaces. By $RU_0^+(G)$ we denote the semigroup of $RO(G)$ consisting of all orthogonal representations admitting the complex structure on their nontrivial summands.

First we introduce the definition of an e-index of a trivial G-space. Using this e-index we measure a homotopical complexity of the fixed point set of G in a most convenient way for our construction of a G-index of the whole space.

3.1. Definition. Suppose that X is a separable metric space. (It is convenient to regard X as a G-space with the trivial action.) Observe that, by the composition of maps, $Z = [S^n, S^n]$ acts on the set of all homotopy classes of maps from X into the sphere S^n, $n \geqslant 1$. Denote by $\Lambda_e(X)$ the set of all nonnegative integers such that

$$[X, S^n] = * \qquad \text{if} \quad G = T^k$$

or

$$[X, S^n] \neq p[X, S^n] \qquad \text{if} \qquad G = Z_p^r.$$

We define the e-index $\gamma_e(X)$ of X as

$$\gamma_e(X) = \inf n, \qquad n \in \Lambda_e(X).$$

(This means that $\gamma_e(X) = \infty$ if $\Lambda_e(X) = \varnothing$.) We also put $\gamma_e(X) = 0$ if $X = \varnothing$, by definition.

As a simple consequence of Definition 3.1 we have the following statements.

3.2. Proposition. *If a separable metric space X has the homotopy type of a CW-complex of dimension n then $\gamma_e(X) \leqslant n$.*

3.3. Proposition. *Suppose that spaces X, Y are of the same homotopy type. Then $\gamma_e(X) = \gamma_e(Y)$.*

3.4. PROPOSITION. $\gamma_e(S^{n-1}) = n$.

The proofs of the above statements are straigtforward.

Now we shall define the notion of a geometrical G-index. For a given $V \in RU_0^+(G)$, the complementing factor of the fixed point space $V^G \subset V$ will be denoted by V_G. For $X \in \mathscr{F}$ we denote by $A_G(X)$ the set of $V \in RU_0^+(G)$ such that there exists a G-map $f: X \to^G V \backslash \{0\}$ satisfying the conditions

$$\dim_R V^G = \gamma_e(X^G) \qquad \text{and} \qquad f^G: X^G \to V^G \backslash \{0\}$$

such that

$$[f] \neq * \qquad \text{in } [X^G, S(V^G)] = [X^G, V^G \backslash \{0\}] \text{ if } G = T^k,$$

and

$$[f] \notin p[X^G, S(V^G)] \qquad \text{if} \quad G = Z_p^r.$$

3.5. DEFINITION. The geometrical G-index of $X \in \mathscr{F}$, denoted by $\gamma_G(X)$, is defined as

$$\gamma_G(X) = \inf(\dim_{\mathbb{R}} V^G + \dim_{\mathbb{C}} V_G), \qquad V \in A_G(X).$$

The relative geometrical G-index of X, denoted by γ_G^0, is defined as

$$\gamma_G^0(X) = \inf(\dim_{\mathbb{C}} V_G,\ V \in A_G(X)).$$

Assume that $A \subset X$, $A = \bar{A}$, is a pair of G-sets belonging to $\mathscr{F}$ such that $X^G \subset A$. By $A_G(X, A)$ we denote the subset of $A_G(X)$ containing all $V \in A_G(X)$ for which there exists a G-map $f: X \to V \backslash \{0\}$ such that $f(A) \subset W \backslash \{0\}$ with $W \subset V$, $W \in RU_0^+(G)$, and $\dim_{\mathbb{C}} W_G = \gamma_G^0(A)$.

3.6. DEFINITION. The geometrical G-index of a pair (X, A) as above, denoted by $\gamma_G(X, A)$, is defined as

$$\gamma_G(X, A) = \inf(\dim_{\mathbb{C}} V_G - \dim_{\mathbb{C}} W_G), \qquad V \in A_G(X, A).$$

Our purpose is to describe simple properties of the geometrical G-index. In the next statement we give a list of basic properties of the G-index we have just defined.

3.7. PROPOSITION. *The geometrical G-indexes γ_G and γ_G^0 have the following properties* (cf. [2, 3, 13]).

1. *Let $A, X \in \mathscr{F}$, $A \subset X$.*

(a) *If* $X^G = \varnothing$ *then* $\gamma_G(X) = \gamma_G^0(X)$.

(b) *If* $X = X^G$ *then* $\gamma_G(X) = \gamma_e(X)$ *and* $\gamma_G^0(X) = 0$.

(c) *If* $A = X^G$ *then* $\gamma_G(X, A) = \gamma_G^0(X)$.

(d) $\gamma_G(X) = \gamma_e(X^G) + \gamma_G^0(X)$.

2. *Assume that* (X, A) *and* (Y, B) *are pairs as in Definition* 3.6. *Suppose that* $\varphi: X \to Y$, $\varphi(A) \subset B$, *is a G-equivariant map such that* $\varphi^G: X^G \to Y^G$ *is a homotopy equivalence. Then*

(a) $\gamma_G(X) \leqslant \gamma_G(Y)$ *and* $\gamma_G^0(X) = \gamma_G^0(Y)$.

(b) *If* $\gamma_G(A) = \gamma_G(B)$ *then* $\gamma_G(X, A) \leqslant \gamma_G(Y, B)$.

3. *Let* (X, A) *be a pair as in Definition* 3.6. *If* $\varphi: X \to Y$ *is a G-equivariant homeomorphism then* $\gamma_G(X) = \gamma_G(Y)$, $\gamma_G^0(X) = \gamma_G^0(Y)$, *and* $\gamma_G(X, A) = \gamma_G(X, \varphi(A))$.

4. *Assume that* X, Y *are closed G-subsets of a G-space* $Z \in \mathscr{F}$. *Then*

(a) *If* $X^G = \varnothing$ *then* $\gamma_G(X \cup Y) \leqslant \gamma_G(X) + \gamma_G(Y)$.

(b) *If* $(\overline{X \backslash Y})^G = \varnothing$ *then* $\gamma_G(\overline{X \backslash Y}) \geqslant \gamma_G(X) - \gamma_G(Y)$.

5. *Assume that a G-set* A, $A = \bar{A} \subset X \in \mathscr{F}$, $X^G \subset A$, *is compact. Then there exists a small* δ*-neighborhood* $N_\delta(A) = \{x \in X: \operatorname{dist}(x, A) \leqslant \delta\}$ *such that* $\gamma_G(N_\delta(A)) = \gamma_G(A)$.

6. *Let* (X, A) *be a pair of G-sets as in* 5. *Then*

$$\gamma_G^0(X \backslash A) = \gamma_G(X \backslash A) \geqslant \gamma_G(X, A) \geqslant \gamma_G(X) - \gamma_G(A).$$

7. *Suppose that* (X, A) *is a pair as in Definition* 3.6 *and* $\gamma_e(X^G) < \infty$. *Then*

(a) *If* X *is a finite dimensional G-metric space with finitely many orbit types then* $\gamma_G(X) < \infty$.

(b) *If* X *is compact then* $\gamma_G(X) < \infty$.

(c) *If* $\gamma_G(X) < \infty$ *then* $\gamma_G(X, A) < \infty$.

8. *Let* $H \subset G$ *be a closed proper subgroup of* G. *Then for the G-space* G/H *we have* $\gamma_G(G/H) = 1$.

9. *Let* (X, A) *be a pair of G-sets as in Definition* 3.6.

(a) *Assume that* $X \backslash A \subset X^H$ *for some closed subgroup* $H \subset G$. *Then* $\gamma_G(X, A) \geqslant 2$ *implies that* $X \backslash A$ *consists of infinitely many orbits.*

(b) *Suppose that* $\gamma_G(X, A) = m$. *Then the set* $X \backslash A$ *contains at least* m *distinct orbits.*

10. *Let* $V \in RU_0^+(G)$ *be a representation of* G. *Then*

$$\gamma_e(S(V^G)) = \dim_{\mathbb{R}} V^G, \qquad \gamma_G(S(V)) = \dim_{\mathbb{R}} V^G + \dim_{\mathbb{C}} V_G,$$

and consequently $\gamma_G^0(S(V)) = \dim_{\mathbb{C}} V_G$.

Proof. A trivial verification shows that 1 follows directly from the definitions. From Proposition 3.3 it follows that $\gamma_e(X^G) = \gamma_e(Y^G)$ under the assumptions of 2. Composing a given $\varphi: X \to^G Y$ with a G-map $f: Y \to V \backslash \{0\}$ satisfying the requirements of the definition of $A_G(Y)$, we see that $A_G(X) \supset A_G(Y)$, which proves 2(a). The same arguments give 2(b). 3 follows directly from 2.

By the definition of γ_G, there exist representations V, $W \in RU_0^+(G)$ and G-maps $f_1: X \to V \backslash \{0\}$, $f_2: Y \to W \backslash \{0\}$ such that $\dim_{\mathbb{R}} V^G = \gamma_e(X^G)$, $\dim_{\mathbb{C}} V_G = \gamma_G^0(X)$, $W^G = \{0\}$, and $\dim_{\mathbb{C}} W = \gamma_G(Y)$, provided $Y^G = \varnothing$. Using the Tietze–Gleason equivariant extension theorem [4, I.2.3], we can extend f and f_2 to G-maps $\bar{f}_1$: $X \cup Y \to^G V$ and $\bar{f}_2: X \cup Y \to^G W$. Putting $f(z) = (\bar{f}_1(z), \bar{f}_2(z))$, we obtain a G-equivariant map $f: X \cup Y \to V \oplus W \backslash \{0\}$, because at least one coordinate of f is different from zero. Note that $(X \cup Y)^G = X^G$ and $(V \oplus W)^G = V^G$. This shows that $V \oplus W \in A_G(X \cup Y)$, which gives 4(a). 4(b) is a consequence of 4(a).

In order to prove 5, observe that from 2(a) we have $\gamma_G(A) \leqslant \gamma_G(N_\delta(A))$ for every $\delta \geqslant 0$. It remains to prove that $\gamma_G(N_\delta(A)) \leqslant \gamma_G(A)$ if δ is sufficiently small. Let $\bar{f}$: $X \to^G V$ be a G-extension of f: $A \to^G V \backslash \{0\}$ given by the Tietze–Gleason theorem. From the compactness of A we conclude that $\bar{f}$ maps $N_\delta(A)$ into $V \backslash \{0\}$ if δ is small enough. This shows that $\gamma_G(N_\delta(A)) \leqslant \gamma_G(A)$, and the proof is complete.

To show 6, take the minimal $V \in A_G(X, A)$ (Def. 3.6). Then $V \in A_G(X)$ and $\gamma_G(X, A) = \dim_{\mathbb{C}} V_G - \dim_{\mathbb{C}} W_G \geqslant \gamma_G^0(X) - \dim_{\mathbb{C}} W_G = \gamma_G^0(X) - \gamma_G^0(A)$, which gives the second inequality of 6. By 4, there exists $\delta > 0$ such that $\gamma_G(N_\delta(A)) = \gamma_G(A)$. Let $f_1: N_\delta(A) \to^G W \backslash \{0\}$ be a G-map with $\dim_{\mathbb{C}} W_G = \gamma_G^0(A)$ and $\dim_{\mathbb{R}} W^G = \gamma_e(A^G) = \gamma_e(X^G)$. For the interior $\mathring{N}_\delta(A)$ of $N_\delta(A)$ we have $X \backslash A \supset X \backslash \mathring{N}_\delta(A)$, and consequently $\gamma_G(X \backslash \mathring{N}_\delta(A)) \leqslant \gamma_G(X \backslash A)$ by 2(a). Next let $f_2: X \backslash \mathring{N}_\delta(A) \to V \backslash \{0\}$ be a G-map with $\dim_{\mathbb{C}} V = \gamma_G(X \backslash N_\delta(A))$, $V^G = \{0\}$. Taking G-extensions $\bar{f}_1$, $\bar{f}_2$ of f_1, and f_2 on X, we can form an equivariant map $f = (\bar{f}_1, \bar{f}_2)$.

By the same argument as in the proof of 4, f maps $X = N_\delta(A) \cup X \backslash \mathring{N}_\delta(A)$ into $V \oplus W \backslash \{0\}$. This shows that $\gamma_G(X, A) = \dim_{\mathbb{C}} V = \gamma_G(X \backslash \mathring{N}_\delta(A)) \leqslant \gamma_G(X - A)$, which gives the first inequality of 6.

In order to prove 7, assume first that $\gamma_e(X^G) < \infty$ and X is finite dimensional. Just as in 4, there exists an open G-set $U \supset X^G$ and a G-map $f_1: U \to V_0 \backslash \{0\}$, where V_0 is a trivial representation of dimension $\dim_{\mathbb{R}} V_0 = \gamma_e(X^G)$. By assumption, the set $X \backslash U$ is a separable finite dimensional G-metric space with finitely many orbit types. It follows from the

Mostow theorem [4, II.10] that there exists a G-embedding $f_2: X \backslash U \to V$, where V is an orthogonal representation of G. Replacing V by its complexification we get a G-embedding into a unitary representation. Moreover, since f_2 is an embedding and $(X \backslash U)^G = \emptyset$, f_2 maps $X \backslash U$ into $V \backslash \{0\}$. As in 4, taking $f = (\bar{f}_1, \bar{f}_2)$: $X = \bar{U} \cup (X \backslash U) \to^G V_0 \oplus V \backslash \{0\}$, we conclude that $V_0 \oplus V \in A_G(X)$, which shows that $A_G(X) \neq \emptyset$. This proves 7(a). The proof of 7(b) is similar. Using the compactness of the G-set $X \backslash U$ (U as in the proof of 7(a)), the existence of tubes [4, II.5], and 4(a), we deduce that $\gamma_G(X \backslash U) < \infty$ (see [2, 3]). The remainder of the proof for 7(b) is as that of 7(a). Similar arguments apply to 7(c), and 7 is proved.

In order to prove 8 we recall that every proper closed subgroup H of the torus T^k can be mapped onto some subgroup of the form $S^1 \times S^1 \times \cdots \times Z_{m_1} \times \cdots \times Z_{m_j}$, $1 \leqslant j \leqslant k$, $m_i \geqslant 1$, by an isomorphism φ of T^k. This implies that $\varphi(H) \cap \{e\} \times \{e\} \times \cdots \times \{e\} \times S^1 = Z_{m_j} \subsetneq S^1$. Let (V, ρ_k) be the one-dimensional complex representation of T^k given by $\rho_k: T^k \to S^1$, $\rho_k(g_1, ..., g_k) = g_k$, and let $(V^{m_j}, \rho_k^{m_j})$ be its m_jth power. The homomorphism $\psi = \rho_k^{m_j} \varphi: T^k \to S^1$ defines a one-dimensional representation of T^k such that $\psi(H) = 1$. Hence ψ maps G/H T^k-equivariantly into S^1, which means that $\gamma_G(G/H) = 1$ as desired in 8. The same idea works in the case $G = Z_p^r$.

To prove 9(a) assume, by contradiction, that X is a disjoint sum $X = A \sqcup \bigsqcup_1^N G/H_i$.

Since $H_i \subset H$, for every i there exists a G-map $\varphi_i: G/H_i \to G/H$. We can form the G-map $\varphi = \bigsqcup_1^N \varphi_i$ from $\bigsqcup_i^N G/H_i$ into G/H. Let

$$f_1: \bigsqcup_1^N G/H_i \to V_1 \backslash \{0\}, \qquad \dim_{\mathbb{C}} V_1 = 1, \qquad V_1^G = \{0\},$$

be the composition of φ with the G-map from G/H into $V_1 \backslash \{0\}$ given by 8. Let next $f_2: A \to V_2 \backslash \{0\}$, $\dim_{\mathbb{R}} V_2^G = \gamma_e(A^G)$, $\dim_{\mathbb{C}} V_G = \gamma_G^0(A)$, be a G-map given by the definition of $\gamma_G(A)$ (Def. 3.1). The nontrivial factor of $V_1 \oplus V_2$ has the complex dimension equal to $\gamma_G^0(A) + 1$. Finally, as in the proof of 4, we can form a G-map $f = (\bar{f}_1, \bar{f}_2)$ from $A \sqcup \bigsqcup_1^N G/H_i$ into $V_1 \oplus V_2 \backslash \{0\}$. This means that $\gamma_G(X, A) = 1$, contrary to the assumption, which proves 9(a). 9(b) follows directly from the inequality $\gamma_G(X, A) \leqslant \gamma_G(X \backslash A)$. Indeed, suppose, by contradiction, that $X \backslash A = \bigsqcup_1^q G/H_i$ with $q < m$. Then $\gamma_G(X \backslash A) = \gamma_G(\bigsqcup_1^q G/H_i) = \sum_1^q \gamma_G(G/H_i) = q$, which contradicts the assumption.

The proof of 10 is based on the Borsuk–Ulam theorem for T^k and Z_p^r (Th. 1 and 2). Let us take $V \in RU_0^+(G) \subset RO^+(G)$. From the definition of γ_e it follows that $\gamma_e(S(V)^G) = \gamma_e(S(V^G)) = \dim_{\mathbb{R}} V^G$, because there are no nontrivial homotopy classes of maps from S^n into S^{n+m} if $m > 0$, and $[S^n, S^n] = Z$ if $n \geqslant 1$. Let $f: S(V) \to S(W)$ be a G-equivariant map as in Definition 3.5. This means that $\dim_{\mathbb{R}} W^G = \dim_{\mathbb{R}} V^G$ and $\deg f \neq 0$ if

$G = T^k$, or $\deg f \not\equiv 0 \pmod p$ if $G = Z_p^r$. From the Borsuk–Ualm theorem we conclude that $\gamma_G(S(V)) \geqslant \dim_{\mathbb{R}} V^G + \dim_{\mathbb{C}} V_G$.

Since the identity $\mathrm{id}: S(V) \to S(V)$ is a G-map $\gamma_G(S(V)) \leqslant \dim_{\mathbb{R}} V^G + \dim_{\mathbb{C}} V_G$. This ends the proof of 10, and the proof of Proposition 3.7 is complete.

Having established the properties of the geometrical G-index, we can now turn to the problem of estimating the number of distinct critical orbits of a G-invariant function.

Let M be a smooth compact manifold without boundary. Assume that G acts smoothly on M and that $f: M \to R$ is a C^1 G-invariant function. The first theorem deals with the case of a fixed point free action of G on M.

THEOREM 3. *Suppose that $M^G = \varnothing$. Then every C^1 G-invariant function $f: M \to R$ has at least $\gamma_G(M)$ distinct critical orbits.*

Proof. The proof uses the standard arguments of the minimax method (see [2, 3, 8, 10–14]).

We first introduce a family of G-subsets of M

$$\Gamma_j = \{X \overset{G}{\hookrightarrow} M; X = \bar{X} \text{ and } \gamma_G(X) \geqslant j\}, \qquad 1 \leqslant j.$$

Observe that $\Gamma_{j+1} \subset \Gamma_j$ and $\Gamma_d \neq \varnothing$ where $d = \gamma_G(M)$. We have also $\Gamma_{d+1} = \Gamma_d$ for every $l > 0$. Next set

$$c_j = \inf_{X \in \Gamma_j} \max_{x \in X} f(x).$$

We thus obtain a sequence of real numbers

$$\min f(x) = c_1 \leqslant c_2 \leqslant \cdots \leqslant c_d = c_{d+1} = \ldots$$

With the above notation we have the following basic statement of the minimax method.

3.8. PROPOSITION. *Let $f: M \to R$ be as in Theorem* 1. *Assume that for some $k \in N$, $k + m - 1 \leqslant d$, we have*

$$c = c_k = c_{k+1} = \cdots = c_{k+m-1}.$$

Let $K_c = \{x \in M: f(x) = c \text{ and } f'(x) = 0\}$. Then $\gamma_G(K_c) \geqslant m$. In particular, c is a critical value of f.

Proof of Proposition 3.8. By definition, K_c is a G-invariant compact set, thus from Proposition 3.7. It follows that there exists $\delta > 0$ such that $\gamma_G(N_\delta(K_c)) = \gamma_G(K_c)$. Using the gradient field of f we get the "equivariant deformation lemma" [2, 12] and the references there). It states that for any

$0 < \varepsilon < \bar{\varepsilon}$ small enough there exists a G-equivariant homeomorphism $\eta: M \to^G M$ such that

(i) $\eta(x) = x$ if $|f(x) - c| > \bar{\varepsilon}_1$,

(ii) $\eta(\{f(x) \leqslant c + \varepsilon\} \backslash N_\delta(K_c)) \subset \{f(x) \leqslant c - \varepsilon_1\}$.

Suppose, by contradiction, that $\gamma_G(K_c) \leqslant m - 1$. By the definition of c_{k+m-1}, there exists $X \in \Gamma_{k+m-1}$ such that $\max_{x \in X} f(x) \leqslant c + \varepsilon$. Let us take a G-invariant closed set $Y = \overline{\eta(X \backslash N_\delta(K_c)}$. From Proposition 3.7. 4(b) and 2 it follows that

$$\gamma_G(Y) \geqslant \gamma_G(X) - \gamma_G(N_\delta(K_c)) > k,$$

because $X \in \Gamma_{k+m-1}$ and $\gamma_G(K_c) \leqslant m - 1$.

On the other hand, $Y \subset \{x \in M: f(x) \leqslant c - \varepsilon\}$ as follows from property (ii) of η. This contradicts the definition of $c_k = c$ and consequently proves Proposition 3.8.

We now complete the proof of Theorem 3. From Proposition 3.8 we know that all the numbers c_j are critical values of f. If $c_1 < c_2 < c_3 \cdots < c_d$ the theorem is proved.

If $c = c_k = c_{k+1} = \cdots = c_{k+m-1}$ then from Proposition 3.8 and property 9(b) of the G-index it follows that K_c contains at least m distinct orbits. Repeating this argument for all multiple c_j, $1 \leqslant j \leqslant d$, we see that f has at least $d = \gamma_G(M)$ distinct critical orbits. This completes the proof of Theorem 1.

The case of a nonempty fixed point set of G on M needs a separate discussion because critical points can cumulate in the fixed point set $M^G \subset M$.

Assume that M, f are as in Theorem 3, but $M^G \neq \varnothing$. First we introduce a family of G-subsets of M:

$$\Gamma_j^0 = X \xhookrightarrow{G} M: M^G \subset X, X = \bar{X} \text{ and } \gamma_G^0(X) \geqslant j\},$$
$$0 \leqslant j \leqslant d^0 = \gamma_G^0(M).$$

Next we define

$$c_j^0 = \inf_{X \in \Gamma_j^0} \max_{x \in X} f(x).$$

Also, let p, $0 \leqslant p \leqslant d^0$, be the number such that

$$c_0^0 = c_1^0 = \cdots = c_p^0 < c_{p+1}^0 \leqslant c_{p+2}^0 \leqslant \cdots$$

(cf. [14]).

Now we can state a theorem estimating the number of distinct critical orbits of a G-invariant function f when the fixed point set of G on M is nonempty.

THEOREM 4. *Every G-invariant C^1-function $f\colon M \to R$ has at least*

$$\mathrm{cat}(M^G) + \gamma_G^0(M) - p$$

distinct critical orbits, where $0 \leqslant p \leqslant \gamma_G^0(M)$ is the number defined above.

The proof of Theorem 2 is similar to that of Theorem 1 and employs the relative index. As a matter of fact, one can follow almost literally the proof of Proposition 5.3 of [14] to show that f has at least $\gamma_G^0(M) - p$ distinct critical orbits in $M \backslash M^G$ (see also [8]). Finally, note that $x \in M^G$ is a critical point of $f^G = f|_{M^G}$ if and only if x is a critical point of f, since f is G-invariant. This gives the proof of Theorem 4.

3.9. *Remark.* Usually, with a view to applications, theorems estimating the number of critical orbits are given for the case of the Palais–Smale functional in a ball, or sphere, of a Hilbert space. After an adaption, the statement of the above theorems carries over to that case. Restricting our considerations to the case of a G-invariant function on a manifold, we demonstrate more strikingly the geometry of the problem discussed.

3.10. *Remark.* Comparing our results (Th. 3, Th. 4, Prop. 3.7) to an analogous theorem of [14] ([14, Prop. 5.3]), we work within a narrow class of groups, admitting $G = T^k$, Z_p^r only. On the other hand, we construct a unique G-index, and moreover we get rid of the assumption of the finiteness of isotropy groups.

REFERENCES

1. A. AMBROSETTI AND G. MANCINI, On a theorem by Ekeland and Lasry concerning the number of periodic Hamiltonian trajectories, *J. Differential Equations* **43** (1981), 1–6.
2. V. BENCI, A geometrical index for the group S^1 and some applications to the research of periodic solutions of ODEs, *Comm. Pure Appl. Math.* **34** (1981), 393–432.
3. H. BERESTYCKI, J.-M. LASRY, G. MANCINI, B. RUF, Existence of multiple periodic orbits on star shaped Hamiltonian surfaces, *Comm. Pure Appl. Math.* **38** (1985), 253–289.
4. G. BREDON, "Introduction to Compact Transformation Groups," Academic Press, San Diego, CA, 1972.
5. K. BORSUK, Drei satze über die n-dimensionale Euklidische Sphare, *Fund. Math.* **21** (1983), 177–190.
6. G. V. COFFMAN, A minimum–maximum principle for a class of nonlinear integral equations, *J. Analyse Math.* **22** (1969), 391–419.
7. T. TOM DIECK, "Transformation Groups and Representation Theory," Springer-Verlag, Heidelberg/New York, 1979.
8. B. DUBROVIN, S. P. NOVIKOV, AND A. T. FOMENKO, "Modern Geometry. Methods of Homology Theory." Science Press, Moscow, 1984. [Russian]

9. I. EKELAND, AND J-M. LASRY, Sur de points critiques de fonctions invariantes par des groupes, *C. R. Acad, Sci. Paris Sér. I Math.* **282** (1976), 841–844.
10. I. EKELAND AND J-M. LASRY, On the number of periodic trajectories for a Hamiltonian flow on a convex energy surface, *Ann. of Math. (2)* **112** (1980), 283–319.
11. E. FADELL AND P. RABINOWITZ, Bifurcation of odd potential operators and an alternative topological index, *J. Funct. Anal.* **26** (1977), 48–67.
12. R. FADELL AND P. RABINOWITZ, Generalized cohomological index theories for Lie groups actions with an application to bifurcation question for Hamiltonian systems, *Invent. Math.* **45** (1978), 139–174.
13. E. FADELL, S. HUSSEINI, AND P. RABINOWITZ, Borsuk–Ulam theorem for arbitrary S^1 actions and applications, *Trans. Amer. Math. Soc.* **274** (1982), 345–359.
14. E. FADELL AND S. HUSSEINI, Relative cohomological index theories, notes, 1984.
15. C. N. LEE AND A. G. WASSERMANN, On the group $JO(G)$, *Mem. Amer. Math. Soc.* **159** (1975).
16. W. MARZANTOWICZ, On the nonlinear elliptic equations with symmetry, *J. Math. Anal. Appl.* **81**, No. 1 (1981), 156–181.
17. L. NIRENBERG, Comments on nonlinear problems, *in* "Conf. Proc., Catania, Sept. 1981."
18. P. RABINOWITZ, Some aspects of nonlinear eigenvalues problems, *Rocky Mountain J. Math.* **3** (1975), 161–202.
19. P. TRACZYK, On the G-homotopy equivalence of spheres of representations MSc. thesis, Dept. of Math., University of Warsaw, 1977. [Polish]
20. P. TRACZYK, On the G-homotopy equivalence of spheres of representations, *Math. Z.* **161** (1978), 257–261.
21. A. WEINSTEIN, Symplectic V-manifolds, periodic orbits of Hamiltonian systems, and the volume of certain Riemannian manifolds, *Comm. Pure Appl. Math.* **30** (1977), 265–271.